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Infinite Groups Generated by Conformal Transformations of Period Two (Involutions and Symmetries).*

BY EDWARD KASNER.

Our main problem is to find all conformal transformations of period 2 and the infinite group generated by such transformations. The result is quite simple and is given in Theorem IV below.

We shall consider only those conformal transformations of the plane which convert the origin into itself and are regular at the origin. Such regular transformations are expressed by power series of the two forms

$$Z = \alpha z + \beta z^2 + \gamma z^3 + \dots, \quad \alpha \neq 0, \quad (1)$$

$$Z = \alpha' z_0 + \beta' z_0^2 + \gamma' z_0^3 + \dots, \quad \alpha' \neq 0, \quad (2)$$

where $z = x + iy$, $z_0 = x - iy$ and the coefficients are arbitrary complex members.† The *direct* (or proper) conformal transformations (1) form a continuous infinite group G ; if we add the *reverse* (or improper) conformal transformations (2), we have a mixed group G' .

It is easy to determine all regular transformations of period 2. In the direct type $Z = f(z)$ the functional equation is $f(f(z)) \equiv z$, that is, $f^2 = 1$; in the reverse type $Z = f(z_0)$ the functional equation is $f(f_0(z)) \equiv z$, that is, $ff_0 = 1$, where f_0 denotes the series whose coefficients are the conjugates of the coefficients of series f . We shall call a transformation the former type (*excluding* the identical transformation) a *conformal involution*, and one of the latter type a *conformal symmetry*.

It can be shown without difficulty (by undetermined coefficients and otherwise) that every involution can be reduced conformally to the simple form $Z = -z$ (symmetry with respect to the origin); and that every symmetry can be reduced to $Z = z_0$ (symmetry with respect to the axis of reals). Conformal symmetry is thus the same as Schwarzian reflexion in an analytic curve.

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† All the work of the present paper is formal; we shall not take into account the convergence of the series. Hence, we are discussing groups of formal power series.

THEOREM I. *The only transformations of period 2 in the general conformal group G' are conformal involutions and conformal symmetries. These are reducible to the normal forms $Z = -z$, $Z = z_0$, respectively.*

The set of all involutions, likewise the set of all symmetries, is in number ∞ . Roughly speaking, the functional equations allow half of the coefficients in the series (1) and (2) to be arbitrary. Neither set by itself, nor the two together, constitute a group.

The question then arises, what types of transformation are obtained by combining any number of involutions or symmetries. The results are given in the following theorems:

THEOREM II. *The only transformations which can be obtained as products of conformal involutions are of the two types*

$$Z = z + \beta z^2 + \gamma z^3 + \dots, \quad \gamma - \beta^2 = 0, \quad (3)$$

$$Z = -z + \beta' z^2 + \gamma' z^3 + \dots, \quad \gamma' + \beta'^2 = 0. \quad (4)$$

These form a mixed group which we denote by G'_{inv} . (the group generated by involutions).

Any one of these transformations can be factored into involutions in an infinitude of ways, of which at least one will contain four or fewer factors. When the number of factors is even we have the type (3), forming a continuous subgroup G_{inv} .

THEOREM III. *The only transformations which can be obtained as products of conformal symmetries are of the two types*

$$Z = \alpha z + \beta z^2 + \gamma z^3 + \dots, \quad |\alpha| = 1, \quad (5)$$

$$Z = \alpha' z_0 + \beta' z_0^2 + \gamma' z_0^3 + \dots, \quad |\alpha'| = 1. \quad (6)$$

These form a mixed group which we denote by G'_{sym} . (the group generated by symmetries).

Any one of these transformations (5), (6) can be factored into symmetries in an infinitude of ways, of which at least one will contain four or fewer factors. In the direct type (5) the number of factors is, of course, always even. This type forms a continuous subgroup G_{sym} .

It will be observed that (3) and (4) are both included as special cases in (5). It follows that every product of any number of involutions can be factored into a product of symmetries. In particular, *every involution can be factored into symmetries*. The converse is obviously not true; for in a symmetry (or product of symmetries) the leading coefficient may be any complex

number whose absolute value is unity, while in the involution case we are restricted to the two values ± 1 .

Every product of symmetries and involutions in any order can be written as a product of symmetries. This gives our

FUNDAMENTAL THEOREM IV. *The group generated by all conformal transformations of period 2 is identical with $G'_{sym.}$. It is represented by (5) and (6), that is, it consists of all regular transformations for which the modulus of the leading coefficient is unity.*

Discussion of Products of Involutions.

The chief question here is to find the form of those transformations which can be obtained as the product of two involutions. Given a function $f(z)$ we inquire whether it is possible to find two functions $\phi(z), \psi(z)$ such that, symbolically,

$$f = \psi\phi, \quad \phi^2 = 1, \quad \psi^2 = 1. \quad (7)$$

The direct treatment by the method of undetermined coefficients is somewhat long and involved. We may simplify the discussion by noticing that every involution $Z = \phi(z), \phi^2 = 1$ can be written in the implicit form

$$Z + z = a(z - Z)^2 + b(z - Z)^4 + c(z - Z)^6 + \dots, \quad (8)$$

where all the coefficients are entirely arbitrary. (In the explicit form (1) half of the coefficients are arbitrary and the others are determined.) The product of two involutions will necessarily have its leading coefficient unity. Our question is to find when the equation

$$Z = z + c_2 z^2 + c_3 z^3 + \dots \quad (9)$$

can result from the elimination of the variable w from the two equations

$$\left. \begin{aligned} z + w &= a(z - w)^2 + b(z - w)^4 + \dots, \\ w + Z &= A(w - Z)^2 + B(w - Z)^4 + \dots, \end{aligned} \right\} \quad (9')$$

defining two arbitrary involutions.

It is convenient to write $c_k = 2^k e_k$ and to introduce

$$t = z - w. \quad (9'')$$

Eliminating z, w, Z from the four equations (9), (9'), (9''), we find the following equation in the single variable t ,

$$\begin{aligned} -t + P + e_2 P^2 + e_3 P^3 + \dots \\ = A \{t + e_2 P^2 + e_3 P^3 + \dots\}^2 + B \{t + \dots\}^4 + \dots, \end{aligned} \quad (10)$$

where

$$P \equiv t + at^2 + bt^4 + ct^6 + \dots$$

Arranging (10) in powers of t and equating corresponding coefficients, we obtain the infinite sequence of equations which must be discussed.

The first two of these equations are

$$a + e_2 = A, \quad 2ae_2 + e_3 = 2Ae_2.$$

This gives a necessary relation $e_3 - 2e_2^2 = 0$, that is, $c_3 - c_2^2 = 0$. We shall not write down the higher equations of the sequence (corresponding to t^3, t^4 , etc.), but merely state that, if we assume $c_2 \neq 0$, there are no further relations between the coefficients of (9). Therefore,

THEOREM V. *Every transformation of the form*

$$Z = z + c_2 z^2 + c_3 z^3 + \dots, \quad c_2 \neq 0, \quad c_3 - c_2^2 = 0 \quad (11)$$

can be factored into two involutions. This can always be done in ∞^1 ways.

Assume next $c_2 = 0$. The first two equations of the sequence give $a = A$, $c_3 = 0$. The equations corresponding to t^4, t^5 , and t^7 now take such a form that we can eliminate a, b and A, B , obtaining a necessary relation

$$4c_4^4 - c_5^3 + 2c_4c_5c_6 - c_4^2c_7 = 0. \quad (12')$$

No additional relations exist if we assume $c_4 \neq 0$. Hence, transformations of the form

$$Z = z + c_4 z^4 + c_5 z^5 + \dots, \quad c_4 \neq 0, \quad (12)$$

where (12') holds, can be factored into two involutions.

A new type arises when $c_4 = 0$, and so on. The final result is

THEOREM VI. *All regular conformal transformations which can be obtained as the product of two involutions are of the form*

$$Z = z + c_{2\kappa} z^{2\kappa} + c_{2\kappa+1} z^{2\kappa+1} + \dots, \quad c_{2\kappa} \neq 0, \quad (13)$$

where $\kappa = 1, 2, 3, \dots$, and a single rational relation

$$R_\kappa(c_{2\kappa}, c_{2\kappa+1}, \dots, c_{4\kappa-1}) = 0 \quad (13')$$

holds between the coefficients.

The form of this relation, as well as the number of coefficients involved, changes with the integer κ , but in all cases $c_{4\kappa-1}$ can be expressed as a rational function of the previous coefficients.

For each value of the integer κ we have a certain set of transformations which can be factored into two involutions. Thus, for $\kappa = 1$, we have the set (11); and for $\kappa = 2$, we have the set defined by (12) and (12'). No one of these sets has the group property. The same is true of the totality of all the sets.

All the transformations in question are, however, included in the larger class,

$$Z = z + c_2 z^2 + c_3 z^3 + \dots, \quad c_3 - c_2^2 = 0, \quad (14)$$

which *does* constitute a group, as may be verified immediately.

A simple example of a transformation of the group (14), which is not of the form specified in Theorem VI, and hence cannot be factored into *two* involutions, is $Z = z + z^4$. We shall now show, however, that *every* transformation of the group (14) can surely be factored into four (or fewer) involutions.

If, in (14), c_2 does not vanish, we already know (Theorem V) that two involutions are sufficient. Consider, therefore, any transformation, T , of our group (14) for which c_2 does vanish. Of course, then, $c_3 = 0$ on account of the relation $c_3 - c_2^2 = 0$. We can factor T into two transformations, T' and T'' , both of the form (14), and such that the coefficients c'_2 and c''_2 do not vanish. This is seen from the fact that the product $T' T''$ is of the form

$$z + (c'_2 + c''_2) z^2 + \dots,$$

and, hence, the coefficient of z^2 can be made to vanish without taking either c'_2 or c''_2 equal to zero. We already know that T' and T'' can each be factored into two involutions. Hence, T can be factored into four involutions.

The type (14) is what we have written as (3) in Theorem II. To complete the proof of this theorem, we merely multiply (11) by a general involution and find the type (4). The latter type can, therefore, surely be factored into three (or fewer) involutions.

Discussion of Products of Symmetries.

A conformal symmetry we have defined as a reverse conformal transformation, $Z = f(z_0)$, whose square is the identical transformation; that is, such that $f(f_0(z)) \equiv z$. The conditions that

$$Z = \alpha z_0 + \beta z_0^2 + \gamma z_0^3 + \dots \quad (15)$$

shall be a symmetry are therefore that the coefficients of the conjugate series

$$Z_0 = \alpha_0 z + \beta_0 z^2 + \gamma_0 z^3 + \dots \quad (15')$$

shall be the same as the coefficients of the series obtained by reverting the original series (15). Therefore,

$$\alpha_0 = \frac{1}{\alpha}, \quad \beta_0 = -\frac{\beta}{\alpha^3}, \quad \gamma_0 = \frac{2\beta^2 - \alpha\gamma}{\alpha^5}, \quad \dots \quad (15'')$$

The first of the conditions gives $|\alpha| = 1$.

We now examine the conditions that a direct transformation (1) shall be expressible as the product of two symmetries. An obviously necessary condition is $|\alpha|=1$. The question is whether transformations of the form

$$Z=\alpha z+\beta z^2+\gamma z^3+\dots, \quad |\alpha|=1, \quad (16)$$

can be factored into symmetries.

Since we have two arbitrary symmetries at our disposal, each involving an infinitude of arbitrary constants, we might, at first sight, expect that the coefficients β, γ, \dots in (16) are entirely arbitrary. But the detailed examination, by the method of undetermined coefficients, shows that this is so only "in general," namely, if we assume that the amplitude θ of the leading coefficient α is irrational; that is, θ/π is an irrational number. This distinction between the *irrational* and the *rational* cases is analogous to that arising in the writer's treatment of another problem of conformal geometry, namely, the invariants of curvilinear (analytic) angles.*

For the irrational case we have this simple result:

THEOREM VII. *Transformations of the form*

$$Z=\alpha z+\beta z^2+\gamma z^3+\dots, \quad (17)$$

where the modulus of α is unity, and the amplitude of α is irrational (so that α is not a root of unity), are always factorable (formally) into two symmetries.

In the rational case such a factoring is *not* always possible. This may be verified most easily by taking the case $\alpha=1$; that is,

$$Z=z+\beta z^2+\gamma z^3+\dots \quad (18)$$

We find, by the method of undetermined coefficients, that there is now a necessary condition on the higher coefficients, namely,

$$|\gamma|-|\beta|^2=0. \quad (18')$$

If this relation does not hold, factoring into two symmetries is impossible. If it does hold, and if $\beta \neq 0$, the factoring is possible. Higher relations arise when $\beta=0$, etc. Analogous results are obtained when α is a root of unity.

Transformations (17), for which α is a root of unity (rational case), are not factorable into two symmetries unless the higher coefficients are subjected to certain restrictions. The form of these higher relations and the number of coefficients involved is different for different rational angles θ .

* See "Conformal Geometry," *Proceedings of the Fifth International Congress of Mathematicians*, Cambridge (1912), Vol. II, pp. 81-87. Also G. A. Pfeiffer, *AMERICAN JOURNAL OF MATHEMATICS*, Vol. XXXVII.

It will not be necessary, however, to employ these higher relations in order to solve our group problem. Transformations factorable into *two* symmetries do not, by themselves, constitute a group. The larger class defined by $|\alpha|=1$ do constitute a group, and we shall show that every one of its transformations can be factored into *four* (or fewer) symmetries.

In the irrational case we already know that two are sufficient. If, now, θ is rational, we may break it up into two parts, θ' and θ'' , both of which are irrational. (This, of course, can be done in an infinitude of ways.) Hence, the given transformation T ,

$$Z = \alpha z + \beta z^2 + \dots, \quad \alpha = e^{i\theta},$$

can be decomposed into two transformations, T' , T'' ,

$$\begin{aligned} Z &= \alpha' z + \beta' z^2 + \dots, & \alpha' &= e^{i\theta'}, \\ Z &= \alpha'' z + \beta'' z^2 + \dots, & \alpha'' &= e^{i\theta''}, \end{aligned}$$

such that T' and T'' both come under the irrational type. (This can be done in ∞ ways, since β', γ', \dots can be taken arbitrarily, β'', γ'', \dots being then determined.) Since, by Theorem VII, we can factor T' and T'' each into two symmetries, it follows that T can be factored into four symmetries.

It follows also that all products with an even number of symmetries as factors make up the continuous group of direct transformations (5). To complete the proof of Theorem III, stated earlier, we need merely observe the reverse type (6) is obtained from (5) by multiplying with the simple symmetry $Z=z_0$. This shows that (6) can be factored into five (or fewer) symmetries. By a simple device we can, however, always reduce the number of factors to three.

If, in the general regular transformation (1) or (2), the leading coefficient α or α' has an absolute value different from unity, then factoring into symmetries or any regular transformations of period 2 will, of course, be impossible. We can, however, reduce the modulus (which expresses the stretching ratio at the origin) to unity by means of a homothetic transformation

$$Z = mz,$$

where m is real. This gives the following interesting result:

Any regular conformal transformation of the plane (converting the origin into itself) can be decomposed into a homothetic transformation together with a finite number (not exceeding four) of conformal symmetries.

That is, it is possible to find four or fewer regular analytic arcs through the origin (analytic elements), such that successive Schwarzian reflexion in these arcs, followed by a stretch, is identical with the arbitrary transformation (1) or (2).

Homothetic transformations combined with involutions generate a continuous subgroup defined by (1) with the single relation $\beta^2 - \alpha\gamma = 0$.

We point out, in conclusion, that our results are valid in the complex (four-dimensional) plane, as well as in the usual real or Gaussian (two-dimensional) plane.*

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* See *Trans. Amer. Math. Soc.*, Vol. XVI (1915), pp. 333-349, for the distinction between real and complex conformal geometry.